

The radiation of a uniformly accelerated charge is beyond the horizon: A simple derivation

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By exploring some elementary consequences of the covariance of Maxwell's equations under general coordinate transformations, we show that even though inertial observers can detect electromagnetic radiation emitted from a uniformly accelerated charge, comoving observers will see only a static electric field. This analysis can add insight into one of the most celebrated paradoxes of the last century.

I. INTRODUCTION

Paradoxes provide good opportunities to learn and teach physics. The long-standing paradox about the electromagnetic radiation emitted by a uniformly accelerated charge has received considerable attention. Eminent figures such as Pauli, Born, Sommerfeld, Schott, von Laue, and many others have contributed to this debate with different answers.¹ The relevant questions we consider are: Does a uniformly accelerated charge actually radiate? In a constant gravitational field should free-falling observers detect any radiation emitted by free-falling charges? Is the equivalence principle valid for such situations?

If the answer to the first question is affirmative, a free-falling charge will radiate according to an observer at rest, because in a constant gravitational field, any particle should move with uniform acceleration. However, an observer falling freely with the charge would observe it at rest and no radiation at all. How can this answer be compatible with an affirmative answer to the first question? Moreover, if the equivalence principle is assumed to be valid, we would conclude that a charged particle at rest on a table should radiate, because for free-falling inertial observers the particle is accelerating. To explain this puzzle, we need to recognize that the concept of radiation has no absolute meaning, and that it depends both on the radiation field and the state of motion of the observer.^{2,3,4,5} This dependence is the main conclusion of a celebrated and long debate, exhaustively presented in the recent series of papers by Eriksen and Grøn,¹ where the reader can find relevant references.

Inertial observers have no doubts about the answer to the first question. They will answer it affirmatively by using special relativity and Maxwell's equations, as it is done in classical electrodynamics texts (see, for example, Ref. 6 whose conventions we adopt). Nevertheless, comoving observers, that is, accelerated observers with respect to whom the charge is at rest, will not detect any radiation because the radiation field is confined to a spacetime region beyond a horizon that they cannot access.⁴ As we will see, uniformly accelerated observers are able, in principle, to detect electromagnetic radiation from an inertial charge. These observations are enough to solve the paradoxes posed by the three questions. The last two conclusions are obtained by considering Maxwell's equations for noninertial observers. We will show that we can conclude that comoving observers have no access to the radiation field of a uniformly accelerated charge. The concept of a horizon emerges naturally in this context. Our approach is inspired by the recent analysis of this problem by Gupta and Padmanabhan⁷ whose result is much stronger. They show how to calculate the radiation field for an arbitrarily moving charge by solving Maxwell's equations in the (noninertial) reference frame where the charge is at rest, and then transforming the electromagnetic field to the inertial reference frame by exploiting the tensorial character of Maxwell's equations. As stressed in Ref. 7, this is a remarkable result because the spatial and time dependences of the radiation field in the inertial frame have to be converted into the geometrical properties of the background metric of the reference frame where the charge is static. We take the converse approach here and will show that it is simple to conclude that the electromagnetic field generated by a uniformly accelerated charge is observed by a comoving observer as a purely electrostatic field.

II. THE HYPERBOLIC MOTION

The speed of light c is the maximum speed that a physical body can attain. Thus the uniformly accelerated motion of a particle should have $|v| \rightarrow c$ as $\tau \rightarrow \pm\infty$, where τ is the proper time as measured by a comoving clock.⁶ It is easy to deduce that a particle moving with constant proper acceleration g along the z direction has a hyperbolic worldline

given by the curve $r^a(\tau)$:

$$ct = r^0(\tau) = \frac{c^2}{g} \sinh \frac{g\tau}{c}, \quad (1a)$$

$$x = r^1(\tau) = 0, \quad (1b)$$

$$y = r^2(\tau) = 0, \quad (1c)$$

$$z = r^3(\tau) = \frac{c^2}{g} \cosh \frac{g\tau}{c}. \quad (1d)$$

There is no loss of generality if the motion is restricted to the z direction. Such a worldline is displayed in Fig. 1.

A. The Horizons

The velocity of a particle according to Eq. (1) approaches $\pm c$ as $\tau \rightarrow \pm\infty$, and its trajectory tends asymptotically to the lines $\pm ct = z$, with $z > 0$ as shown in Fig. 1. Consider the point Q . Its past light-cone intersects the hyperbolic trajectory. Indeed, a large part of the trajectory ($\tau < \tau_{\text{ret}}^Q$) is entirely contained inside its past light-cone, implying that the point Q could be causally influenced by signals emitted by the particle for $\tau \leq \tau_{\text{ret}}^Q$; no signal emitted for $\tau > \tau_{\text{ret}}^Q$ will influence Q , because the points of the trajectory with $\tau > \tau_{\text{ret}}^Q$ are not contained in the past light-cone of Q . Moreover, a signal emitted in the space time Q will affect only the region corresponding to its future light-cone, implying that no signal emitted in Q will reach the particle moving according to Eq. (1). The line $ct = z$ acts as a *future event horizon* for regions I and IV, or, equivalently, a *past event horizon* for regions II and III. No signal emitted in regions II or III will reach regions I and IV, although signals emitted in I or IV can cross the line and enter into regions II and III. Analogously, the line $-ct = z$ is a future event horizon for regions III and IV, or a past event horizon for I and II.

Because the hyperbolic trajectory is entirely contained in region I, the lines $ct = z$ and $-ct = z$ act, respectively, as the future and past horizons for a particle under uniformly accelerated motion. We will see that such structures appear naturally when we consider the radiation emitted by a uniformly accelerated charged particle.

B. The Radiation in the inertial frame

The metric of the inertial Minkowski spacetime is given by

$$ds^2 = \eta_{ab} dx^a dx^b = c^2 dt^2 - dx^2 - dy^2 - dz^2, \quad (2)$$

where $x^a = (ct, x, y, z)$. Maxwell's equations are not only Lorentz invariant, they can also be cast in a generally covariant way, valid for any reference frame with the metric G_{ab} ,

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0, \quad (3)$$

$$\frac{1}{\sqrt{G}} \partial_a (\sqrt{G} F^{ab}) = \frac{4\pi}{c} J^b, \quad (4)$$

where $G = |\det G_{ab}|$ and J^b is the external 4-current.^{6,8} Equation (3) is automatically satisfied if the 4-potential A_b is introduced: $F_{bc} = \partial_b A_c - \partial_c A_b$. In the inertial Minkowski frame, the radiation emitted by a uniformly accelerated charge e corresponds to the solution of Eqs. (3) and (4) with $G_{ab} = \eta_{ab}$, $J^a(x) = ec \int d\tau V^a(\tau) \delta^{(4)}(x - r(\tau))$, and $V^a = \dot{r}^a$. Such a solution is given by:⁶

$$F^{ab} = e \left[\frac{1}{V^c(x_c - r_c)} \frac{d}{d\tau} \frac{(x^a - r^a)V^b - (x^b - r^b)V^a}{V^c(x_c - r_c)} \right]_{\text{ret}}, \quad (5)$$

where the quantity between the brackets is to be evaluated at the retarded time τ_{ret} given by (see Fig. 1):

$$[x^a - r^a(\tau_{\text{ret}})][x_a - r_a(\tau_{\text{ret}})] = \left(ct - \frac{c^2}{g} \sinh \frac{g\tau_{\text{ret}}}{c} \right)^2 - \rho^2 - \left(z - \frac{c^2}{g} \cosh \frac{g\tau_{\text{ret}}}{c} \right)^2 = 0, \quad (6)$$

with $\rho^2 = x^2 + y^2$, leading to

$$z \cosh \frac{g\tau_{\text{ret}}}{c} - ct \sinh \frac{g\tau_{\text{ret}}}{c} = \frac{g}{2} \left(\frac{\rho^2}{c^2} + \frac{z^2}{c^2} - t^2 + \frac{c^2}{g^2} \right). \quad (7)$$

In the inertial frame we can read from F^{ab} the usual three-dimensional components of the electric and magnetic field as

$$F^{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \quad (8)$$

If we use the equation $V^a(x_a - r_a) = c(ct \cosh \frac{g\tau}{c} - z \sinh \frac{g\tau}{c})$, we obtain after some straightforward algebra,

$$\frac{1}{xz} E_x = \frac{1}{yz} E_y = \frac{4}{g^2 \left(\frac{\rho^2}{c^2} + \frac{z^2}{c^2} - t^2 \right)^2 - \frac{c^4}{g^2}} E_z \quad (9a)$$

$$\frac{1}{ctx} B_y = -\frac{1}{cty} B_x = \frac{eg}{c^2 \left(ct \cosh \frac{g\tau_{\text{ret}}}{c} - z \sinh \frac{g\tau_{\text{ret}}}{c} \right)^3}, \quad (9b)$$

and $B_z = 0$, where Eq. (7) was explicitly used in the expression for E_z . These are the electromagnetic fields due to a uniformly accelerated charge moving according to Eq. (1).

The radiation content can be extracted by separating the components that drop off as $1/R$ from the usual Coulomb $1/R^2$ fields.⁶ As shown in Fig. 1, only regions I and II can experience the fields in Eq. (9). The main conclusion of Ref. 4 is that, even though radiation components are present in both regions, only observations performed in region II (and, perhaps, also on the boundary $ct = z$ between regions I and II) would allow us to detect unambiguously the radiation emitted by the charge. This conclusion implies that the comoving observer would not detect any radiation at all, because region II is inaccessible to uniformly accelerated observers.⁴ Although this conclusion is correct, its logical derivation is involved and not intuitive. Two regions of spacetime which have qualitatively distinct behavior for the radiation field according to inertial observers are identified and, then, it is shown that the comoving observers have access only to the region where inertial observers are not able to detect any radiation field.⁴ However, this conclusion does not directly imply that the comoving observers cannot detect the radiation because, as we have discussed, the detection of radiation has no absolute meaning because the detection depends both on the radiation field and the state of motion of the observer.

C. No Radiation in the comoving frame

We can show directly that a comoving observer will observe the fields in Eq. (9) as a static electric field. The reference frame of a uniformly accelerated observer corresponds to the Rindler spacetime,⁸ which in our case is spanned by the coordinates $x'^a(x) = (c\tau(t, z), x, y, \xi(t, z))$ defined by

$$t = \sqrt{\frac{2\xi}{g}} \sinh \frac{g\tau}{c} \quad (10a)$$

$$z = c \sqrt{\frac{2\xi}{g}} \cosh \frac{g\tau}{c}, \quad (10b)$$

with $\xi > 0$. The particle under the hyperbolic motion (1) in the Rindler reference frame is at rest at $\xi = c^2/2g$, and its proper time is measured by τ . In these coordinates the spacetime interval is given by

$$ds^2 = G_{ab} dx^a dx^b = 2g\xi d\tau^2 - dx^2 - dy^2 - c^2 \frac{d\xi^2}{2g\xi}. \quad (11)$$

Note that the coordinates defined by Eq. (10) cover only region I of the original Minkowski spacetime. Because static observers (ξ is a constant) correspond to uniformly accelerating observers in the original Minkowski spacetime, their velocity in the inertial frame will approach c as $\tau \rightarrow \infty$, implying that no signal coming from region II will ever reach

them (see Fig. 1). As mentioned, the line $ct = z$ behaves as an event horizon, and no signal emitted in regions II or III can escape into regions I and IV.

The coordinate transformation (10) can be used to obtain the solution of Maxwell's equations (3) and (4) for the Rindler spacetime with a charge e at rest in $\xi = c^2/2g$. Recall that the electromagnetic field F^{ab} is a tensor and hence under a coordinate transformation $x^a \rightarrow x'^a(x)$ it transforms as^{6,8}

$$F'^{ab} = \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} F^{cd}. \quad (12)$$

Because Maxwell's equations (3) and (4) are covariant under general coordinate transformations, F'^{ab} will be a solution for the coordinate system $x'^a(x)$ if F^{ab} is a solution of Eqs. (3) and (4) in the coordinate system x^a . The magnetic components of F'^{ab} are given by

$$F'^{13} = \frac{1}{c} \frac{\partial \xi}{\partial t} E_x + \frac{\partial \xi}{\partial z} B_y, \quad (13a)$$

$$F'^{23} = \frac{1}{c} \frac{\partial \xi}{\partial t} E_y - \frac{\partial \xi}{\partial z} B_x, \quad (13b)$$

$$F'^{12} = B_z = 0. \quad (13c)$$

Strictly speaking, we need to be careful about the interpretation of F'^{13} , F'^{23} , and F'^{12} as the components of the magnetic field as observed in the Rindler reference frame. A proper definition of electric and magnetic fields for noninertial reference frames can be obtained from the Lorentz force formula.⁷ This issue will not be relevant to our analysis.

By using Eq. (9), we have

$$\frac{1}{y} F'^{13} = \frac{1}{x} F'^{23} = \frac{eg}{c^2 \left(ct \cosh \frac{g\tau_{\text{ret}}}{c} - z \sinh \frac{g\tau_{\text{ret}}}{c} \right)^3} \left(\frac{1}{c} \frac{\partial \xi}{\partial t} z + c \frac{\partial \xi}{\partial z} t \right). \quad (14)$$

From the transformation (10), we can calculate

$$\frac{\partial \tau}{\partial t} = \frac{z}{2\xi}, \quad \frac{\partial \tau}{\partial z} = -\frac{t}{2\xi}, \quad (15a)$$

$$\frac{\partial \xi}{\partial t} = -gt, \quad \frac{\partial \xi}{\partial z} = \frac{gz}{c^2}, \quad (15b)$$

leading to $F'^{13} = F'^{23} = 0$. Therefore the only nonvanishing components of the electromagnetic field experienced by comoving observers are F'^{01} , F'^{02} , and F'^{03} . Because the only nonvanishing component of the 4-current J'^a is J'^0 for a static charge in the Rindler reference frame, we conclude from Eq. (4) that the remaining nonvanishing components of the electromagnetic field are static, that is, $\partial_0 F'^{01} = \partial_0 F'^{02} = \partial_0 F'^{03} = 0$, so that there is no radiation field in region I, the Rindler reference frame.

This result answers our question. A comoving observer will not detect any radiation from a uniformly accelerated charge. The comoving observer can receive signals only from regions I and IV. The field emitted by the accelerated charge does not reach region IV, and in region I, it is interpreted by the comoving observer as a static field. We note that essentially the same argument was used by Rohrlich to show that in a static homogeneous gravitational field, static observers do not detect any radiation from static charges.³

The situation is qualitatively different beyond the horizon in region II. Although uniformly accelerated observers will never receive any information from region II, they can affect this region. The coordinate system (10) can be extended to include region II by considering $\xi < 0$ and

$$t = \sqrt{\frac{-2\xi}{g}} \cosh \frac{g\tau}{c} \quad (16a)$$

$$z = c \sqrt{\frac{-2\xi}{g}} \sinh \frac{g\tau}{c}. \quad (16b)$$

The metric (11) and the expressions (15), valid for region I, also hold in region II, but with a crucial difference due to the change of sign of the metric components: in region II, ξ instead of τ plays the role of a time parameter.

Thus the metric (11) is not static in region II. The magnetic components of F'^{ab} in region II can be obtained from transformations such as Eq. (13) if we take into account that the components 0 (temporal) and 3 (spatial) are, respectively, ξ and $c\tau$:

$$\frac{1}{y}F'^{13} = \frac{1}{x}F'^{23} = \frac{eg}{c^2 \left(ct \cosh \frac{g\tau_{\text{ret}}}{c} - z \sinh \frac{g\tau_{\text{ret}}}{c} \right)^3} \left(\frac{\partial\tau}{\partial t} z + c^2 \frac{\partial\tau}{\partial z} t \right) \quad (17a)$$

$$= -\frac{e}{\left(ct \cosh \frac{g\tau_{\text{ret}}}{c} - z \sinh \frac{g\tau_{\text{ret}}}{c} \right)^3}. \quad (17b)$$

The fields (17), together with the electric components that can be obtained in an analogous way, are time-dependent solutions of the (vacuum) Maxwell equations in region II, having radiating parts.¹¹ However, they are inaccessible to a comoving observer because they are confined beyond his/her future horizon.

III. CONCLUDING REMARKS

The physics of the Rindler space is sufficiently subtle to deserve some extra remarks. Trajectories with constant ξ (see Fig. 2) correspond to uniformly accelerated trajectories in the inertial frame, but with *distinct* accelerations. The trajectory (1) corresponds to the static worldline $\xi = c^2/2g$ in the Rindler frame. From the inertial frame point of view, the true comoving observer should correspond also to $\xi = c^2/2g$, because any other static Rindler observer would be in relative motion according to the inertial frame point of view with respect to the charge following Eq. (1). Our results show that the electromagnetic field of the uniformly accelerated charge is realized as a purely electrostatic field everywhere in the Rindler frame, implying that even observers with $\xi \neq c^2/2g$, for which the charge is indeed accelerating when observed from the inertial point of view, would not detect the emitted radiation. This observation, which anticipates an intriguing quantum result described by Matsas,⁹ reinforces the role played by the horizon, the unique property that the trajectories of these distinct observers have in common. (see Fig. 2)

The discussion can be considerably enriched by the introductions of quantum mechanical concepts. The classical radiation emitted by the accelerated charge in the inertial frame consists of a large number of real photons, which due to some subtle quantum effects cannot be detected by comoving observers.¹⁰ To illustrate the novelties brought by quantum mechanics, consider in the Minkowski space a uniformly accelerated observer following a trajectory such as that in Eq. (1) and a charge at rest at the origin. The worldline for this charge is the ct axis, and it is restricted to regions II and IV. The solution of Maxwell equations in the inertial frame is the static Coulomb field

$$\frac{1}{x}E_x = \frac{1}{y}E_y = \frac{1}{z}E_z = \frac{e}{(x^2 + y^2 + z^2)^{3/2}}, \quad (18)$$

which spreads over all four regions of Fig. 1. In region I, where a uniformly accelerated observer can detect any field coming from the charge, the static Coulomb field will be measured by such observers as a time-depending electromagnetic field with components

$$\begin{aligned} F'^{01} &= \frac{z}{2\xi}E_x, & F'^{02} &= \frac{z}{2\xi}E_y, & F'^{03} &= E_z \\ F'^{31} &= -\frac{gt}{c}E_x, & F'^{32} &= -\frac{gt}{c}E_y, & F'^{12} &= 0. \end{aligned} \quad (19)$$

The fields (19) are solutions of the (vacuum) Maxwell equations in region I as seen by uniformly accelerated observers. These fields have radiative components, although it is not so easy in this case to identify the terms dropping off as $1/R$.¹¹ Note that the accelerated observers are completely unaware of the charge fate in region II. Because they can detect only the contributions coming from regions I and IV, they will never discover what eventually happens to the charge in region II if it, for instance, accelerates or even if it vanishes.

An analysis of this problem based on quantum field theory, however, demands that the trajectory of the charge be entirely inside region I. Is it possible to conclude something in this case? Astonishing the answer is yes.⁹ In the Rindler reference frame static trajectories with $\xi \rightarrow \infty$ (see Fig. 2) correspond to uniformly accelerated trajectories in the inertial frame, restricted by construction to region I, but with proper acceleration $g = c^2/2\xi \rightarrow 0$. Thus, they correspond to inertial trajectories! Now we can answer the question of whether uniformly accelerated observers could detect any photon emitted by these specific inertial charges, and the answer is no.⁹

The detection of radiation is not the only paradox involving accelerated charges. Another very interesting paradox is related to the radiation reaction force. As we discussed, an inertial observer detects the radiation emitted by a

uniformly accelerated charge. He/she can even calculate the (nonvanishing) total radiated power. But we know from classical electrodynamics that the radiation reaction force vanishes for a constant proper acceleration.⁶ Hence, what is acting as the source of the radiated power? How is it possible to conserve energy in this case? Interesting questions, but that's another story ...¹²

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Figure Captions

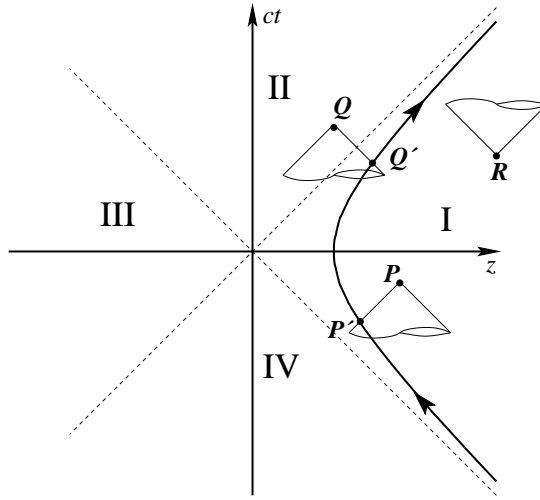


FIG. 1: The hyperbolic trajectory $r^a(\tau)$ given by Eq. (1). The retarded time τ_{ret} associated with a given point (ct, x, y, z) corresponds to the (unique) intersection of the past light-cone of (ct, x, y, z) with the trajectory $r^a(\tau)$. For instance, $Q' = (c^2/g)(\sinh(g\tau_{\text{ret}}^Q/c), 0, 0, \cosh(g\tau_{\text{ret}}^Q/c))$ and $P' = (c^2/g)(\sinh(g\tau_{\text{ret}}^P/c), 0, 0, \cosh(g\tau_{\text{ret}}^P/c))$ define, respectively, the retarded times τ_{ret}^Q and τ_{ret}^P associated with the points Q and P . The future light-cone is the boundary of the causal future of a given point. Thus, any event occurring, for instance, in the spacetime point R will affect only the region enclosed by its future light-cone, with the light-cone surface reserved only to signals moving with velocity c . Note that only regions I and II are affected by the fields due to a charged particle with a worldline given by Eq. (1).

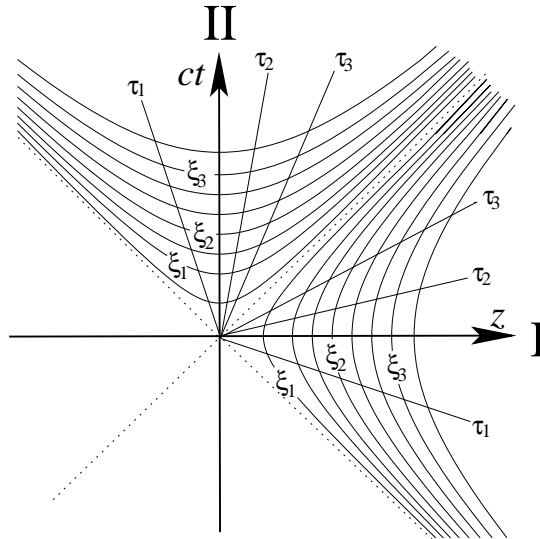


FIG. 2: The lines of constant ξ and τ according to Eqs. (10) and (16), respectively, for the regions I and II. In region I, the Rindler frame where $\xi > 0$, the identified lines correspond to $\xi_1 < \xi_2 < \xi_3$ and $\tau_1 < \tau_2 < \tau_3$. Lines of constant ξ (the hyperbola) are timelike. On the other hand, for region II, known as the Milne frame where $\xi < 0$, the lines of constant τ are timelike. The identified lines in II correspond to the cases $\tau_1 < \tau_2 < \tau_3$ and $\xi_1 > \xi_2 > \xi_3$. The horizon, the boundary $ct = z$ between I and II, corresponds to one half of the degenerated hyperbola corresponding to $\xi = 0$.